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Horospheres on abelian covers

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—To Ricardo Mañé, in memoriam.

Abstract. We consider the strong stable foliation of the geodesic flow for a noncompact, connected abelian cover of a closed negatively curved manifold. We show that there exists proper leaves, and that non-proper leaves are dense.

0. Introduction

Let (M, g) be a closed Riemannian manifold, with negative sectional curvature and infinite first homology group, e.g. a hyperbolic closed Riemann surface. We consider $\pi : \bar{M} \mapsto M$ a regular connected cover of M such that the covering group is abelian and infinite, and $\bar{\Phi}_t : T^1\bar{M} \mapsto T^1\bar{M}, t \in \mathbb{R}$, the geodesic flow on the unit tangent bundle of $\bar{M}$. We are interested in the strong stable foliation of $\bar{\Phi}_t$. Leaves of this foliation are defined by

$$W^{ss}(\bar{x}) = \{\bar{y} : d(\bar{\Phi}_t\bar{y}, \bar{\Phi}_t\bar{x}) \to 0 \; as \; t \to +\infty\}.$$

These leaves are smoothly embedded euclidean spaces, and depend continuously on the point $\bar{x}$ [Anosov 69]. This foliation was particularly studied in the case of surfaces of constant negative curvature, when the foliation is given by the orbits of the horocycle flow. In this case, it is known that the strong stable foliation is transitive [Hedlund 36], ergodic for the Haar measure [Babillot-Ledrappier 96]. In general, it is known that the strong stable foliation is never uniquely ergodic [BL96]. Here we show that the strong stable foliation is not minimal, in the sense that there exist leaves which are proper submanifolds in $T^1\bar{M}$, but almost,

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in the sense that a leaf which is not proper is dense. More precisely, we have:

Theorem A. *Let $T^1\bar{M}$ be the unit tangent bundle of a noncompact connected abelian cover of M. Then there exist points $\bar{x}$ in $T^1\bar{M}$ such that the strong stable leaf of $\bar{x}$ is proper.*

Theorem B. *Let $T^1\bar{M}$ be the unit tangent bundle of a connected $\mathbb{Z}^d$ cover of M. Then a strong stable leaf is either dense or proper.*

Observe that for a general Riemann surface, there are examples where the horocycle flow is transitive on the unit tangent bundle to the surface, but where there exist horocycles which are neither dense nor proper ([Starkov 95]).

We obtain equivalent results when considering the action of $\bar{\Gamma} = \pi_1(\bar{M})$ on the space of horospheres. Namely let $\bar{\pi} : \tilde{M} \mapsto \bar{M}$ be the universal cover of $\bar{M}$ (and therefore of M), and fix a point o in $\tilde{M}$. Then the Busemann function $B : \tilde{M} \times \partial\tilde{M} \mapsto \mathbb{R}$ is defined by:

$$B(p,\xi) = \lim_{t\to+\infty} d(p, q_\xi(t)) - t,$$

where $q_\xi(t), t \in \mathbb{R}$ is the geodesic in $\tilde{M}$ such that $q_\xi(0) = o, q_\xi(+\infty) = \xi$. The space of strong stable leaves is homeomorphic to the space $\partial\tilde{M} \times \mathbb{R}$ and the action of $\bar{\Gamma}$ is given by $\gamma(\xi, t) = (\gamma\xi, t - B(\gamma^{-1}o, \xi))$ (see e.g. [Babillot 96]); properties of this action reflect properties of the strong stable foliation in the quotient space $\bar{\Gamma} \setminus T^1\tilde{M} = T^1\bar{M}$. In order to state the corresponding results, we say that a group $\bar{\Gamma}$ of isometries of $\tilde{M}$ is an abelian cover group (respectively a $\mathbb{Z}^d$ cover group), if there is a torsionless group Γ of isometries of $\tilde{M}$ with compact quotient such that $\bar{\Gamma}$ is a normal subgroup of Γ, with abelian infinite quotient (respectively with quotient $\mathbb{Z}^d$). We then have:

Theorem A'. *Let $\bar{\Gamma}$ be an abelian cover group of isometries of $\tilde{M}$. Then there are points ξ in $\partial\tilde{M}$ such that the set $\{(\gamma\xi, -B(\gamma^{-1}o, \xi)), \gamma \in \bar{\Gamma}\}$ is discrete in $\partial\tilde{M} \times \mathbb{R}$.*

Theorem B'. *Let $\bar{\Gamma}$ be a $\mathbb{Z}^d$ cover group of isometries of $\tilde{M}$, ξ in $\partial\tilde{M}$. Then, the set $\{(\gamma\xi, -B(\gamma^{-1}o, \xi)), \gamma \in \bar{\Gamma}\}$ is either discrete or dense in $\partial\tilde{M} \times \mathbb{R}$.*

In order to prove Theorems A' and B', recall that a point ξ in $\partial\tilde{M}$ is called horospherical for $\bar{\Gamma}$ if $\inf_{\gamma\in\bar{\Gamma}} B(\gamma o, \xi) = -\infty$. For an abelian cover group, this definition does not depend of the reference point o. We then have:

Proposition 1'. *Let $\bar{\Gamma}$ be an abelian cover group of isometries of $\tilde{M}$. Then there exist points which are not horospherical.*

Proposition 2'. *Let $\bar{\Gamma}$ be a $\mathbb{Z}^d$ cover group of isometries of $\tilde{M}$. Then a point ξ is horospherical if, and only if, the set $\{(\gamma\xi, -B(\gamma^{-1}o, \xi)), \gamma \in \bar{\Gamma}\}$ is dense in $\partial\tilde{M} \times \mathbb{R}$.*

Proposition 3'. *Let $\bar{\Gamma}$ be a group of isometries of $\tilde{M}$ such that*

$$\inf\{d(p, \gamma p); p \in \tilde{M}, \gamma \in \bar{\Gamma}\} \geq d$$

for some positive d, ξ a non-horospherical point. Then,

$$\{(\gamma\xi, -B(\gamma^{-1}o, \xi)), \gamma \in \bar{\Gamma}\}$$

is a discrete subset in $\partial\tilde{M} \times \mathbb{R}$.

Theorems A' and B' follow immediately from propositions 1', 2' and 3'. In the case of surfaces of constant curvature -1, the action of $\bar{\Gamma}$ on the space of horospheres can also be identified with the linear action on $\mathbb{R}^2 \setminus 0$ of the corresponding group of (2×2) matrices. Observe that 0 is a fixed point for the linear action of matrices, and that a direction in $\mathbb{R}^2$ correspond to a horospherical point if and only if 0 is adherent to the orbit of a vector in that direction. Our arguments also yield:

Theorems A" & B". *Let $\bar{\Gamma}$ be an abelian cover group of (2×2) matrices, acting linearly on $\mathbb{R}^2$. The orbit of a vector in $\mathbb{R}^2$ is either dense or discrete. There are infinite discrete orbits.*

In the following sections, we state and prove the counterpart of Propositions 1', 2' and 3' pertaining to the strong stable foliation. We also state the corresponding Propositions 1" and 2". Proposition 1 follows from properties of ω-minimizing geodesics ([Bangert 90], [Mather 91]). Proposition 2' for Fuchsian groups of the first kind is due to Hedlund [H36]. Once we know that the strong stable foliation is transitive (and this follows from [BL96], see below Proposition 2), Hedlund's proof can be extended to our case. Finally, Proposition 3 rests on a simple geometric argument. The properties stated above for abelian cover groups are in fact stable by finite quotient. Therefore we may (and will in the rest of the paper) assume that the covering group $\bar{\Gamma} \setminus \Gamma$ is isomorphic to $\mathbb{Z}^d$, for some positive d.

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1. Minimizing geodesics

In this section, we consider $\bar{\pi} : \tilde{M} \mapsto \bar{M}$ and $\pi : \bar{M} \mapsto M$, where M is a closed manifold, $\tilde{M}$ the universal cover of M, $\bar{M}$ a regular cover of M. Denote $\bar{\Gamma}, \Gamma$ the covering groups associated respectively to $\bar{\pi}$ and $\pi \circ \bar{\pi}$; we assume that $\bar{\Gamma} \setminus \Gamma$ is isomorphic to $\mathbb{Z}^d$, for some positive d. Denote $\mathcal{H}$ the Hurewicz homomorphism $\mathcal{H} : \pi_1(M) \mapsto H_1(M, \mathbb{Z})$, $\bar{H}_1 = \mathcal{H}(\bar{\Gamma})$, and $\bar{H}^1$ the subspace of $H^1(M, \mathbb{R})$ which annihilates $\bar{H}_1$ in the duality between real homology and cohomology. Then, $\dim \bar{H}^1 = d$.

Fix a Riemannian metric on M, and consider the geodesic flow $\tilde{\Phi}_t$ (respectively $\bar{\Phi}_t, \Phi_t$), $t \in \mathbb{R}$ on the unit tangent bundle $T^1\tilde{M}$ (respectively $T^1\bar{M}, T^1M$). Let $\mathcal{M}$ be the set of Φ-invariant Borel probability measures on T^1M. For a smooth closed 1-form ω on M, the function $\omega : TM \mapsto \mathbb{R}$ associates to a point $x = (p, v)$, the value of the form ω_p on the vector v of T_pM. Define:

$$k(\omega) = \max\{\int_{T^1M} \omega d\mu; \mu \in \mathcal{M}\}.$$

Observe that, if α is a smooth function, $k(\omega + d\alpha) = k(\omega)$, so that k defines a functional on $H^1(M, \mathbb{R})$. This functional is a norm, dual to the stable norm (see e.g. [GLP, chapter 4] for the definition of the stable norm and [Massart] for its expression in terms of invariant probability measures). In particular there exists $\eta \in \bar{H}^1$ such that $k(\eta) = 1$. In the rest of this section we choose and fix such an η. By compactness of $\mathcal{M}$, the set $\mathcal{M}_\eta$ of measures μ satisfying $\int \omega d\mu = 1 = \max\{\int_{T^1M} \omega d\mu; \mu \in \mathcal{M}\}$, for some ω representing η, is a closed convex set and a face in $\mathcal{M}$; in particular, there exist ergodic probability measures in $\mathcal{M}_\eta$. We also choose and fix $\mu \in \mathcal{M}_\eta$.

Proposition 1. *Choose η in $\bar{H}^1$ such that $k(\eta) = 1$, μ ergodic in $\mathcal{M}_\eta$, x a point in the support of μ (i.e. every neighborhood of x in T^1M has positive μ-measure) and $\bar{x}$ in $T^1\bar{M}$ such that $D\pi\bar{x} = x$. Let $\bar{p}(t), t \in \mathbb{R}$, be the geodesic in $\bar{M}$ with initial condition $\bar{x}$. Then, the geodesic $\bar{p}(t)$ minimizes distances in $\bar{M}$ between its points, i.e. we have, for any s, t in $\mathbb{R}$:*

$$d_{\bar{M}}(\bar{p}(s), \bar{p}(t)) = |s - t|.$$

Proposition 1 directly follows from [M91, Proposition 3] if, after having chosen a closed 1-form ω for representing η, we consider the Lagrangian L on TM defined by:

$$L(p, v) = \frac{1}{2}\|v\|^2 - \omega_p(v).$$

The Euler-Lagrange flow of L is the geodesic flow on TM. In particular, the energy $E(p, v) = \frac{1}{2}\|v\|^2$ is invariant and for any $a \geq 0$, the energy level $T^aM = \{(p, v) : \|v\|^2 = a^2\}$ is invariant. The spherical bundle T^1M corresponds to the energy $\frac{1}{2}$. For a probability measure μ on TM invariant under the geodesic flow, the action $A(\mu)$ is defined by $A(\mu) = \int L d\mu$, and the Aubry-Mather constant of the Lagrangian L is defined as minus the infimum of the actions of invariant probability measures on TM. We have:

Lemma 1. *For the above Lagrangian L, the Aubry-Mather constant is $\frac{1}{2}$. The action minimizing probability measures are the measures supported on T^1M such that $1 = \int_{T^1M} \omega d\mu$.*

Proof. Since $\mu \mapsto A(\mu)$ is a linear functional, we may define the Aubry-Mather constant C as minus the infimum of the actions of ergodic measures. But for an ergodic measure, $\|v\|^2$ is constant, so that C is given by:

$$-C = \inf_{a \geq 0} \{\frac{1}{2}a^2 - \max\{\int \omega d\mu; \mu \in \mathcal{M}_a\}\},$$

where $\mathcal{M}_a$ is the set of ergodic probability measures on TM which are supported on T^aM. Multiplication by the positive real a defines a natural bijection between $\mathcal{M}_1$ and $\mathcal{M}_a$, so that:

$$\begin{aligned} -C &= \inf_{a\geq 0}\{\frac{1}{2}a^2 - a\max\{\int \omega d\mu; \mu \in \mathcal{M}_1\}\} \\ &= \inf_{a\geq 0}\{\frac{1}{2}a^2 - ak(\omega)\} \\ &= \inf_{a\geq 0}\{\frac{1}{2}a^2 - a\} = -\frac{1}{2} \end{aligned}$$

Moreover, the measures for which the minimum is achieved are the ergodic probability measures μ which satisfy $\mu \in \mathcal{M}_1$ and $1 = \int_{T^1M} \omega d\mu$. By averaging over the ergodic decomposition, we see that any invariant probability measure with action $-\frac{1}{2}$ is supported by T^1M and maximizes $\int \omega d\mu$.

We now prove Proposition 1. Let i be the canonical embedding of T^1M in TM. By lemma 1, the measure $i_*(\mu)$ is an action minimizing ergodic probability measure on TM. The support of the measure $i_*(\mu)$ is i(support μ). In particular, the point $i(x)$ belongs to the support of the measure $i_*(\mu)$. The argument of [M91, page 184/185] shows that the geodesic in $T\bar{M}$ with initial condition $\bar{x}$ minimizes the action of $\bar{L} = L \circ D\pi + \frac{1}{2}$ between any two of its points. Observe now that since $\eta \in \bar{H}^1$, the 1-form $\omega \circ D\pi$ is exact on $\bar{M}$. This means that when a curve $c(t), t \in [0, T]$ minimizes $\bar{L}$ between its endpoints, the curve has to minimize the integral

$$\int_0^T (\frac{1}{2}\|\dot{c}(t)\|^2 + \frac{1}{2})\, dt$$

over curves joining its endpoints. Moreover, since the minimizing curve is a speed-one geodesic, it minimizes the same integral among curves parametrized by arc length with the same endpoints. For such curves, the integral is the length of the curve. Thus, the geodesic starting from $\bar{x}$ realizes the distance between its points.

To finish this section, we verify that Proposition 1 implies Proposition 1' of the Introduction. Choose $\bar{x}$ given by Proposition 1, and $\tilde{x}$ such that $D\bar{\pi}\tilde{x} = \bar{x}$. The geodesic $\tilde{p}(t)$ in $\tilde{M}$ starting from $\tilde{x}$ has the property that for any positive t, $\tilde{p}(t)$ is closest to $\tilde{p}(0)$ than to any other point of the orbit of $\tilde{p}(0)$ under $\bar{\Gamma}$. In other words, for any $\gamma \in \bar{\Gamma}$, any $t \geq 0$, we have $d(\gamma\tilde{p}(0), \tilde{p}(t)) \geq t$.

Assume now that the metric on M has nonpositive curvature and let ξ be the point at infinity of the geodesic ray $\tilde{p}(t), t \geq 0$, then for any $\gamma \in \bar{\Gamma}$, we have:

$$B(\gamma\tilde{p}(0), \xi) = \lim_{t\to+\infty} (d(\gamma\tilde{p}(0), \tilde{p}(t)) - t) \geq 0.$$

The point ξ is not horospherical.

In the same way, we get:

Proposition 1". *Let $\bar{\Gamma}$ be an abelian cover group of (2×2) matrices, acting linearly on $\mathbb{R}^2$. Then, there exists $v \in \mathbb{R}^2$ such that 0 is not adherent to the orbit $\bar{\Gamma}v$.*

2. Dense horospheres

We keep the setting and the notations of Section 1, and we furthermore assume that the metric on M has negative curvature. In order to fix notations, we put on T^1M, $T^1\bar{M}$ the restriction of the Sasaki metric on TTM, the one which makes horizontal and vertical projections on TM isometries with orthogonal kernels. We first have:

Proposition 2. *Let M be a closed Riemannian manifold with negative curvature, $\bar{M}$ a connected $\mathbb{Z}^d$ cover of M. The strong stable foliation is transitive on $T^1\bar{M}$: there exists a point $\bar{x}$ with $W^{ss}(\bar{x})$ dense in $T^1\bar{M}$.*

The proof of Proposition 2 closely follows the scheme of the proof of ergodicity of the horocycle flow on $T^1\bar{M}$, given in [BL96] in the case of surfaces, taking care of the fact that, since the leaves are not in general the trajectories of a flow, some formulations have to be slightly modified. Namely choose a countable family $\{\mathcal{O}_n\}$ of open sets in $T^1\bar{M}$ and $\bar{\lambda}$ the product of the Lebesgue measure on strong stable manifolds in $T^1\bar{M}$ and of the lift of the Margulis measure to the transversals to the strong stable foliation on $T^1\bar{M}$. It suffices to show that for each n, the set of points $\bar{x}$ for which $W^{ss}(\bar{x})$ do not intersect $\mathcal{O}_n$ has $\bar{\lambda}$ measure 0. Fix n for the rest of the proof.

We are going to apply Theorem 3 of [BL96], so we choose suitable parameters in this Theorem as stated (the trusting reader can go directly to next paragraph; for the more cautious reader, we follow the notations of [BL96], Section 2). We indeed are in the setting of [BL96], since the geodesic flow is a mixing Anosov flow on T^1M and since, as observed by [Pollicott & Sharp], for any connected $\mathbb{Z}^d$ cover, Assumption (A) is satisfied (see [PS94], Proposition 7). So we first choose a regular fundamental domain D of the action of $\mathbb{Z}^d$ on $T^1\bar{M}$ such that the closure of the open set $\mathcal{O}_n$ lies inside D. Then we choose τ small enough that there exists a set $\bar{A}$ with closure in $\mathcal{O}_n$ which projects on T^1M into a set A belonging to $\mathcal{A}_\tau$, and that there is a positive measure set E of centers of balls of $\mathcal{B}_\tau$ disjoint from the projection of the boundary of D ($\mathcal{A}_\tau$ and $\mathcal{B}_\tau$ are collections of the subsets of T^1M which are small disks of radius τ on the strong unstable and strong stable leaves respectively). Finally choose $\delta > 0$ such that the set $\bar{A}_\delta = \cup_{-\delta\leq t\leq\delta}\bar{\Phi}_t\bar{A}$ is still contained in $\mathcal{O}_n$ and we shall take $g = \chi_{(-\delta,\delta)}$. Observe that [BL96] Corollaries 3 and 5 apply in the case of the Margulis measure (which is the case $v = 0$), so that, for $\bar{\lambda}$-almost every point $\bar{x}$, there exists a sequence $\{T_k(\bar{x}), k \geq 0\}$ of larger and larger times such that for all k, $\pi(\bar{\Phi}_{T_k}(\bar{x})) \in E$ and that $\lim_{k\to\infty} b_k/T_k = 0$, where $b_k \in \mathbb{Z}^d$ are such that $\bar{\Phi}_{T_k}(\bar{x}) \in b_kD$. We apply [BL96] Theorem 3 with these choices of D, τ, $g = \chi_{(-\delta,\delta)}$, $\pi A \in \mathcal{A}_\tau$, $B^{ss}(\bar{\pi}\Phi_{T_k}(\bar{x}), \tau) \in \mathcal{B}_\tau$, $b = b_k$ and times $T_k(\bar{x})$.

We thus obtain for $\bar{\lambda}$-almost every point $\bar{x} \in T^1\bar{M}$ an estimate on some sum over the geodesic paths of length $T_k(\bar{x})$ going from $\bar{A}_\delta \subset \mathcal{O}_n$ to a strong stable disk of radius τ around $\bar{\Phi}_{T_k}\bar{x}$, and this estimate goes to infinity with T_k. This means that one can find some T such that there is at least one geodesic path of length T going from $\mathcal{O}_n$ to a point in $W^{ss}(\bar{\Phi}_T(\bar{x}))$. Therefore there is at least one point in the intersection of $\mathcal{O}_n$ and $\bar{\Phi}_{-T}W^{ss}(\bar{\Phi}_T(\bar{x})) \subset W^{ss}(\bar{x})$. This proves Proposition 2.

Corollary. *Let $\bar{\Gamma}$ be an $\mathbb{Z}^d$ cover group of isometries of $\tilde{M}$. Then the action of $\bar{\Gamma}$ on $\partial\tilde{M} \times \mathbb{R}$ is transitive.*

In fact, Proposition 2 tells that there is a point $\tilde{x}$ in $T^1\tilde{M}$ such that $\cup_{\gamma\in\bar{\Gamma}}\gamma W^{ss}(\bar{x})$ is dense in $T^1\tilde{M}$. The conclusion follows.

The proof of Proposition 2' from the above Corollary now follows the steps and the arguments of Hedlund; for the convenience of the reader, we shall sketch this proof, referring to [H36] for details. Firstly, it is clear that if $\bar{\Gamma}(\xi, 0)$ is dense, then the point ξ is horospherical. Conversely, we know that there is some ξ_0 with $\bar{\Gamma}(\xi_0, 0)$ dense, and also with $\bar{\Gamma}(\xi_0, t)$ dense for any $t \in \mathbb{R}$ (shifting by t is a homeomorphism which commutes with the action of $\bar{\Gamma}$).

Lemma 2. *Assume ξ is one of the two fixed points of a hyperbolic $\gamma \in \bar{\Gamma}$. Then $\bar{\Gamma}(\xi, 0)$ is dense in $\partial\tilde{M} \times \mathbb{R}$.*

In fact, we first send ξ arbitrarily close to ξ_0 by some γ_i and we can choose integers j_i such that

$$B(\gamma^{-j_i}\gamma_i^{-1}o, \xi) = B(\gamma^{-j_i}o, \xi) + B(\gamma_i^{-1}o, \xi)$$

is bounded independently of i. Then there is a subsequence $\{\gamma_i'\}$ of group elements $\gamma_i' = \gamma_i.\gamma^{j_i}$ and u in $\mathbb{R}$ such that $\{\gamma_i'(\xi, 0)\}$ converge towards (ξ_0, u) as $i \to \infty$. Therefore $\bar{\Gamma}(\xi, 0)$ is dense in $\partial\tilde{M} \times \mathbb{R}$ as well.

If ξ is a horospherical point, there are γ_i such that $\{B(\gamma_i o, \xi)\}$ go to $-\infty$ and $\{\gamma_i^{-1}\xi\}$ converge to some point ω, as $i \to \infty$. Choose a hyperbolic $\gamma \in \bar{\Gamma}$, the fixed points of which $\gamma^{\pm}$ are distinct from ω. Then one can choose the powers k_i, $|k_i| \to \infty$ so that

$$B(\gamma_i\gamma^{k_i}o, \xi) = B(\gamma_i o, \xi) + B(\gamma^{k_i}o, \gamma_i^{-1}\xi)$$

is bounded, independently of i. (Observe that $B(\gamma^k o, \gamma_i^{-1}\xi)$ varies from 0 to $+\infty$ with $|k|$, with bounded increments). From this, it follows that $\{\gamma^{-k_i}\gamma_i^{-1}(\xi, 0)\}$ approaches some point $(\gamma^{\pm}, u)$, which has a dense orbit by Lemma 2. Thus $\bar{\Gamma}(\xi, 0)$ is dense, as claimed.

For the sake of completeness, we recall (a particular case of [Greenberg 63]):

Proposition 2". *Let $\bar{\Gamma}$ be an abelian cover group of (2×2) matrices, acting linearly on $\mathbb{R}^2$ and assume that $\bar{\Gamma}$ acts transitively on directions. If, for $v \in \mathbb{R}^2$, 0 is adherent to $\bar{\Gamma}v$, then $\bar{\Gamma}v$ is dense in $\mathbb{R}^2$.*

3. Proper horospheres

In this section, we only assume that the negatively curved manifold $\bar{M}$ has bounded geometry, that is, there exist positive constants a, b

and d such that the sectional curvature K on $\bar{M}$ satisfies everywhere $-b^2 \leq K \leq -a^2$ and such that the injectivity radius of $\bar{M}$ is at least d. We still denote $\bar{\Gamma} = \pi_1(\bar{M})$ and $\tilde{M}$ the universal cover of $\bar{M}$. We are interested in this Section in cases where the strong stable leaf W^{ss} might be a properly immersed submanifold of $T^1\bar{M}$. This happens if, and only if, the set $\bar{\Gamma}W^{ss}$ is a closed subset of $T^1\tilde{M}$, i.e. if, and only if, the orbit under $\bar{\Gamma}$ of W^{ss} in the space of horospheres is a closed subset of $\partial\tilde{M} \times \mathbb{R}$, i.e. if, and only if, this orbit is discrete.

Definition. A geodesic $\bar{p} = \bar{p}(t), t \in \mathbb{R}$, in $\bar{M}$ is called asymptotically almost minimizing if it is has speed 1 and if for any positive δ there is T such that for any s, t in $\mathbb{R}$, $s, t \geq T$, we have : $d_{\bar{M}}(\bar{p}(s), \bar{p}(t)) \geq |s-t| - \delta$.

The statements in the Introduction all directly follow from the results of the previous Sections and from the two following Propositions:

Proposition 3. *Let $\bar{M}$ be a negatively curved manifold with bounded geometry, $\bar{p}$ an asymptotically almost minimizing geodesic, $\bar{x}(t) \in T^1\bar{M}$ the derivative vector at $\bar{p}(t)$. Then for each $t \in \mathbb{R}$, the strong stable leaf $W^{ss}(\bar{x}(t))$ is proper.*

Proposition 4. *Let $\bar{M}$ be a nonpositively curved manifold, $\bar{\pi} : \tilde{M} \to \bar{M}$ its universal cover, $\tilde{p}$ a geodesic in $\tilde{M}$. If the point $\tilde{p}(+\infty)$ is not horospherical for $\pi_1(\bar{M})$, then the geodesic $\bar{\pi}\tilde{p}$ is asymptotically almost minimizing.*

Proof of Proposition 3. Observe that if $\bar{p}$ is an asymptotically almost minimizing geodesic, if $\bar{x}(t) \in T^1\bar{M}$ is the derivative vector at $\bar{p}(t)$ and if $\bar{y} \in W^{ss}(\bar{x}(t))$, then the geodesic with initial vector $\bar{y}$ is asymptotically almost minimizing as well. Therefore it suffices to show that the strong stable leaf $W^{ss}(\bar{x}(t))$ do not accumulate on itself at $\bar{x}(t)$, or in other words, that there exists a positive number ε such that if the point $\bar{y}$ satisfies $\bar{y} \in W^{ss}(\bar{x}(t))$ and $d_{T^1\bar{M}}(\bar{y}, \bar{x}(t)) \leq \varepsilon$, then $\bar{y}$ belongs to a relatively compact neighborhood of $\bar{x}(t)$ for the topology of the leaf $W^{ss}(\bar{x}(t))$ defined by the induced metric. Without loss of generality, we assume $t = 0$ and write $\bar{x}$ for $\bar{x}(0)$. The proof is based on:

Lemma 3. *Let $\bar{x}$ be the initial condition of an asymptotically almost* *minimizing geodesic $\bar{p}$. For any positive η, there exists a positive ε such that if the point $\bar{y}$ satisfies $\bar{y} \in W^{ss}(\bar{x})$ and $d_{T^1\bar{M}}(\bar{y}, \bar{x}) < \varepsilon$, then for all positive t,*

$$d_{\bar{M}}(\bar{q}(t), \bar{p}(\mathbb{R}^+)) < \eta,$$

where $\bar{q}(t), t \in \mathbb{R}$, is the geodesic in $\bar{M}$ with initial condition $\bar{y}$.

Proof. If the points $\bar{x}$ and $\bar{y}$ are closed, then $d_{\bar{M}}(\bar{q}(t), \bar{p}(\mathbb{R}^+))$ is small for some interval of time containing 0. Let t_0 be the first positive time when $d_{\bar{M}}(\bar{q}(t), \bar{p}(\mathbb{R}^+))$ reaches $\inf(\eta, d/2)$. We shall find ε such that $d_{T^1\bar{M}}(\bar{y}, \bar{x}) < \varepsilon$ and the finiteness of t_0 lead to a contradiction.

Fix $\delta(\eta)$, consider the corresponding T in the definition of asymptotically almost minimizing geodesic, and choose ε_0 so small that if $d_{T^1\bar{M}}(\bar{y}, \bar{x}) \leq \varepsilon_0$, then $d_{T^1\bar{M}}(\bar{q}(T), \bar{p}(T)) \leq \delta$ and $t_0 \geq T + t_1(\eta)$. Let P be the point $\bar{p}(T)$, $Q = \bar{q}(t_0)$, P' the point of $\bar{p}((T, +\infty))$ closest to Q, $P'' = \bar{p}(t_2(\eta))$ where $t_2(\eta)$ is very large. Note t'_0 such that $P' = \bar{p}(t'_0)$. We shall consider the two right angle triangles $PP'Q$ and $P''P'Q$ in $\bar{M}$. We have $P'Q = \inf(\eta, d/2)$, $PQ \leq \varepsilon_0 + t_0 - T$, $P''Q \leq d_{\bar{M}}(Q, \bar{q}(t_2)) + d_{\bar{M}}(P'', \bar{q}(t_2)) \leq t_2 - t_0 + \delta(\eta)$, if $t_2(\eta) \geq t_0 + t_1$ is large enough that $d_{\bar{M}}(\bar{p}(t_2), \bar{q}(t_2)) \leq \delta(\eta)$. On the other hand, we know from the minimizing property that $PP' \geq t'_0 - T - \delta$, $P'P'' \geq t_2 - t'_0 - \delta$.

The curvature being smaller than $-a^2$, the two comparaison triangles in the space of constant curvature $-a^2$ have larger angles. Applying Cosine Rule I to the comparaison triangles, we get:

$$\cosh aPQ \geq \cosh aPP' \cosh aP'Q,$$
$$\cosh aP''Q \geq \cosh aP''P' \cosh aP'Q.$$

Therefore,

$$\cosh a(\varepsilon_0 + t_0 - T). \cosh a(t_2 - t_0 + \delta) \geq$$
$$\geq \cosh a(t'_0 - T - \delta). \cosh a(t_2 - t'_0 - \delta).(\cosh a(\inf(\eta, d/2)))^2.$$

If t_1 and t_2 are large enough, we obtain:

$$\exp a(t_2 - T + \delta + \varepsilon_0) \geq \exp a(t_2 - T - 3\delta) \ . \ (\cosh a(\inf(\eta, d/2)))^2,$$

a contradiction if δ and ε_0 are sufficiently small.

We now prove Proposition 3. Take $\bar{y} \in W^{ss}(\bar{x})$ such that $d_{T^1\bar{M}}(\bar{y}, \bar{x})$ is very small. By Lemma 3, $d_{\bar{M}}(\bar{q}(t), \bar{p}(\mathbb{R}^+))$ is small for all positive t, in particular smaller than $d/2$. Therefore, the function $d_{\bar{M}}(\bar{q}(t), \bar{p}(\mathbb{R}^+))$ is a convex function of t, decreasing to 0 as $t \to \infty$. Thus the function $d_{T^1\bar{M}}(\bar{y}(t), \bar{x}(t'(t)))$ is also small and going to 0 as $t \to \infty$, where $t'(t)$ is such that $\bar{p}(t'(t))$ is the point of $\bar{p}(\mathbb{R}^+)$ closest to $\bar{q}(t)$. This shows that $\bar{y}$ lies in the local stable manifold of $\bar{x}$. Since $\bar{y}$ lies in the global strong stable manifold of $\bar{x}$, $\bar{y}$ lies in the local strong stable manifold of $\bar{x}$.

Proof of Proposition 4. Consider a geodesic $\tilde{p}$ in $\tilde{M}$ such that the point $\xi = \tilde{p}(+\infty)$ is not horospherical, i.e. $B(\gamma o, \tilde{p}(+\infty))$ is bounded away from $-\infty$, uniformly for $\gamma \in \pi_1(\bar{M})$. Given a positive δ, one can find $\bar{\gamma}$ such that setting $o' = \bar{\gamma} o$, we have for the Busemann function B' defined with reference point o': $B'(\gamma o', \tilde{p}(+\infty)) \geq \delta/4$ for all $\gamma \in \pi_1(\bar{M})$. In particular, the geodesic $\bar{q}$ defined by $\bar{q} = \bar{\pi}\tilde{q}$, where $\tilde{q}(0) = o'$ and $\tilde{q}(+\infty) = \xi$ satisfies, for all positive t :

$$t - \delta/4 \ \leq \ d_{\bar{M}}(\bar{q}(0), \bar{q}(t)) \ \leq \ t.$$

The geodesic $\bar{q}$ is $\delta/2$-minimizing, and the geodesic $\bar{\pi}\tilde{p}$, which is asymptotic to $\bar{q}$, is eventually δ-minimizing.

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